

# Probability and distributions

# Probability: How likely is something to happen?



<sup>1</sup>Image: Smith, C. 2013. (Public domain).

## Frequentist interpretation of probability

How frequently was the bus late after 120 morning rides?

- ▶ Late: 24 times
- ▶ Not late: 96 times

Given these observation frequencies, predict the probability that it will be late on any given morning

## Frequentist interpretation of probability

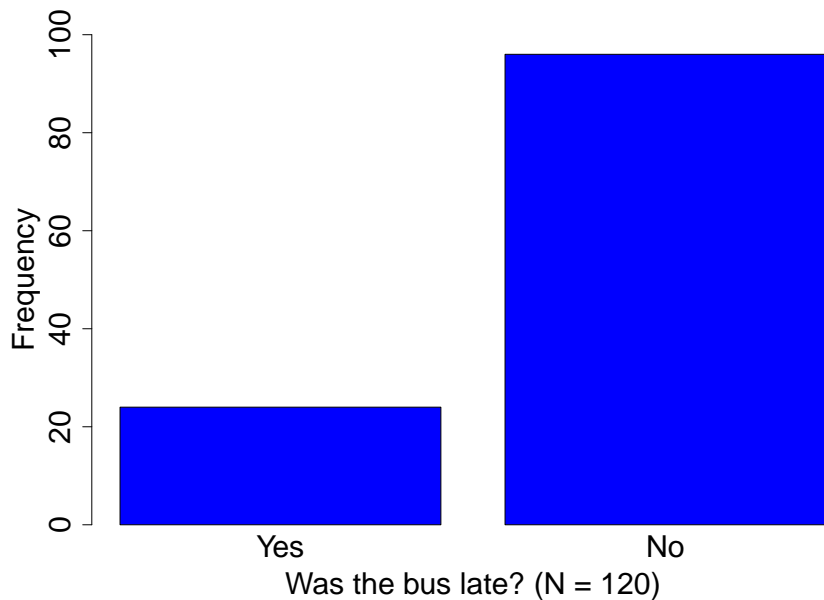
How frequently was the bus late after 120 morning rides?

- ▶ Late: 24 times
- ▶ Not late: 96 times

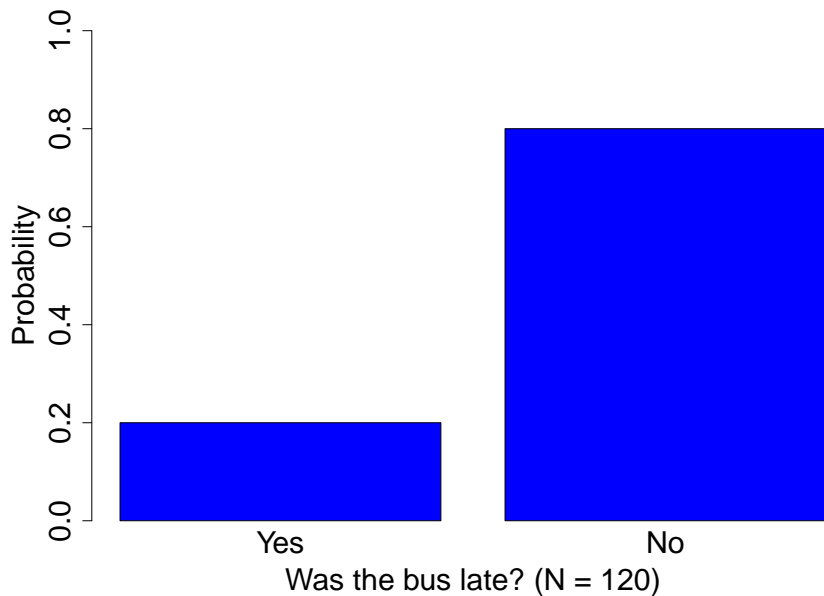
**Probability is the long-term *relative* frequency**

$$Pr(Late) = \frac{24}{24 + 96} = \frac{24}{120} = 0.2$$

## Frequentist interpretation of probability



## Frequentist interpretation of probability



## Frequentist interpretation of probability

How frequently was the bus late after 120 morning rides?

- ▶ Late: 24 times  $\rightarrow Pr(Late) = 0.2$
- ▶ Not late: 96 times  $\rightarrow Pr(Not\ late) = 0.8$

Values **must** sum to 1:

- ▶  $Pr(Event) = 0$ : An event *never* occurs
- ▶  $Pr(Event) = 1$ : An event *always* occurs

## Frequentist interpretation of probability

How frequently was the bus late after 120 morning rides?

▶ Late: 24 times  $\rightarrow Pr(Late) = 0.2$

▶ Not late: 96 times  $\rightarrow Pr(Not\ late) = 0.8$

$$Pr(outcome) = \frac{\textit{Times outcome occurs}}{\textit{Number of trials}}$$

## What can we predict?

- ▶ Extreme climate events
- ▶ Genotypes of offspring
- ▶ Risk of disease spread
- ▶ Population extinction
- ▶ Outcomes in games

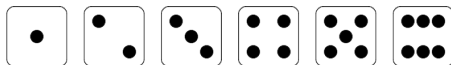
## Games of chance: Probability is a key component









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<sup>1</sup>Image: Korsalka, Z. 2021. (Public domain).

## Rolling two dice; what can happen?















	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

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<sup>1</sup>Image: All dice images are public domain.

## Probability of rolling two 1s













						
	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

$$\Pr(Dice_1 = 1 \ \& \ Dice_2 = 1) = \frac{1}{36} = 0.0278$$

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<sup>1</sup>Image: All dice images are public domain.

## Probability of rolling at least one 4













						
	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

$$\Pr(a\ four) = \frac{11}{36} = 0.3056$$

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<sup>1</sup>Image: All dice images are public domain.

## Probability of rolling **exactly** one 4

						
	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

$$\Pr(\text{exactly one four}) = \frac{10}{36} = 0.2778$$

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<sup>1</sup>**Image:** All dice images are public domain.

## Adding or multiplying probabilities

**Mutually exclusive:** Events cannot happen simultaneously

- ▶ Cannot roll 2 values with one dice
- ▶ You can roll a 1 **or** a 2, not both

**Not mutually exclusive:** Events can happen simultaneously

- ▶ Can roll 2 values with two dice
- ▶ You can roll a 1 **and** another 1

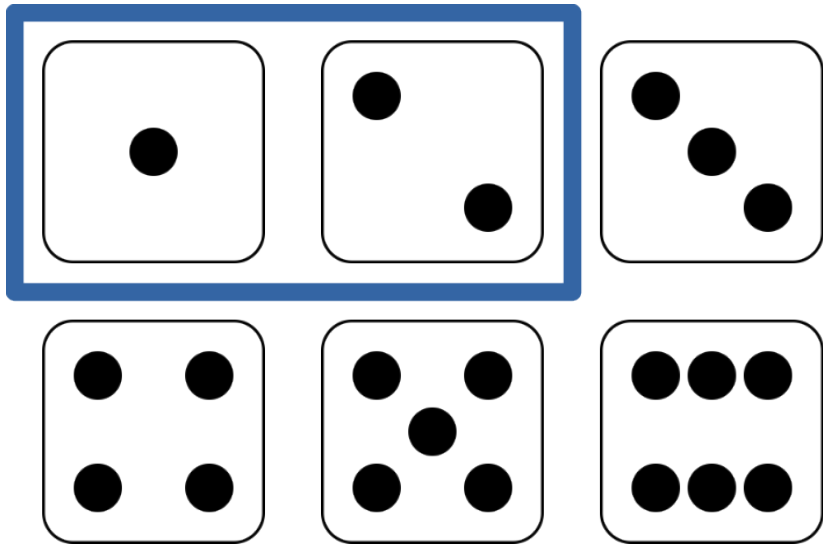
## Adding or multiplying probabilities

**Mutually exclusive:** Events cannot happen simultaneously

- ▶ Roll a 1 **or** a 2 with one dice
- ▶  $\Pr(\text{Roll} = 1 \text{ or } \text{Roll} = 2)$
- ▶  $\Pr(\text{Roll} = 1) + \Pr(\text{Roll} = 2)$

$$\Pr(\text{one or two}) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = 0.3333$$

## Adding or multiplying probabilities



## Adding or multiplying probabilities

- ▶ Add probabilities when events are mutually exclusive
- ▶ Multiply probabilities when events are **not** mutually exclusive













## Adding or multiplying probabilities

**Not mutually exclusive:** Events can happen simultaneously

- ▶ Can roll 2 values with two dice
- ▶ You can roll a 1 **and** another 1
- ▶  $\Pr(\text{Roll}_1 = 1 \text{ and } \text{Roll}_2 = 1)$
- ▶  $\Pr(\text{Roll}_1 = 1) \times \Pr(\text{Roll}_2 = 1)$

$$\Pr(\text{one and one}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} = 0.0278$$

## Probability of rolling a 1 and another 1













						
	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

$$\Pr(\text{one and one}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} = 0.0278$$

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<sup>1</sup>Image: All dice images are public domain.

## Probability of rolling two 1s **or** two 6s













						
	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

$$\Pr(\text{two 1s} \mid \text{two 6s}) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36} = 0.0556$$

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<sup>1</sup>Image: All dice images are public domain.













## Probability of rolling a 1 and a 2

						
	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

$$\Pr(\text{one \& two}) = \left(\frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36} = 0.0556$$

<sup>1</sup>Image: All dice images are public domain.

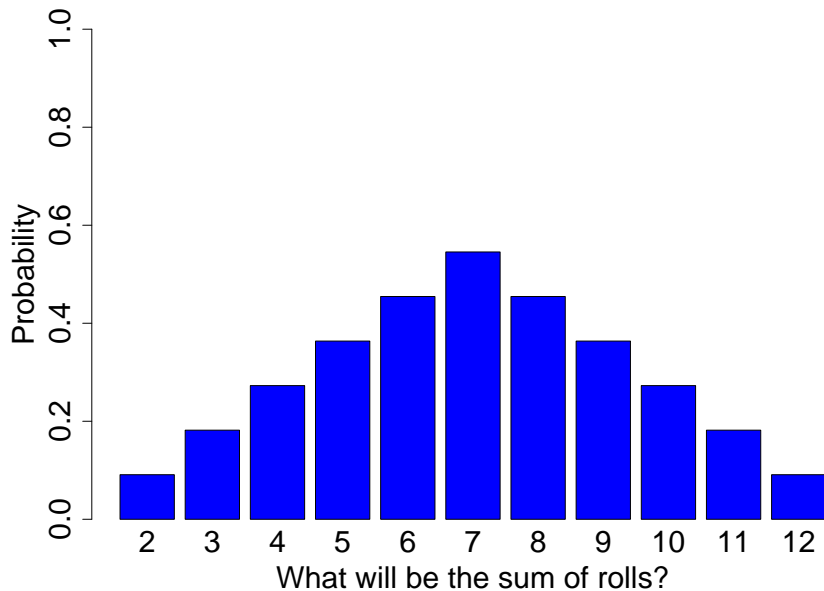
## Sum of rolls: what's the distribution?

						
	2	3	4	5	6	7
	3	4	5	6	7	8
	4	5	6	7	8	9
	5	6	7	8	9	10
	6	7	8	9	10	11
	7	8	9	10	11	12

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<sup>1</sup>Image: All dice images are public domain.

Sum of rolls: what's the distribution?



## Sampling with and without replacement

- ▶ **With replacement:**

Whatever is sampled gets put back before sampling again

- ▶ **Without replacement:**

Whatever is sampled doesn't get put back before sampling again

## Sampling with and without replacement



Pick two cards: probability you sample 2 aces with versus without replacement?

**With replacement:**

$$\frac{2}{10} \times \frac{2}{10} = 0.04$$

**Without replacement**

$$\frac{2}{10} \times \frac{1}{9} = 0.0222$$

Event is an individual **trial**

- ▶ Successful:  $\Pr(X)$
- ▶ Unsuccessful:  $1 - \Pr(X)$

Successes over  $N$  trials

- ▶ Binomial distribution
- ▶ Coin flips heads

## Probability distributions: Binomial distribution

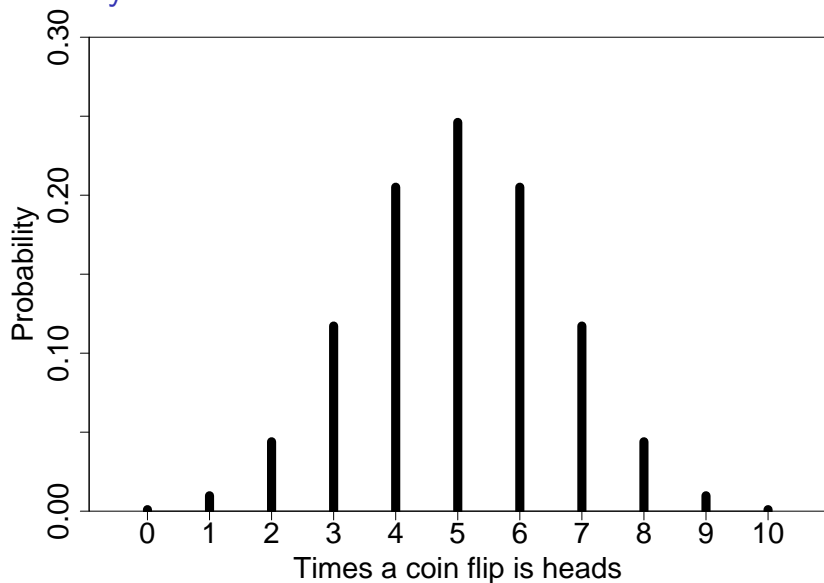


Figure 1: Probability of times a coin flips heads in 10 trials.

## Probability distributions: Poisson distribution

How many observations over a period of time?



Also applies to a period of space

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<sup>1</sup>Image: [Public domain](#).

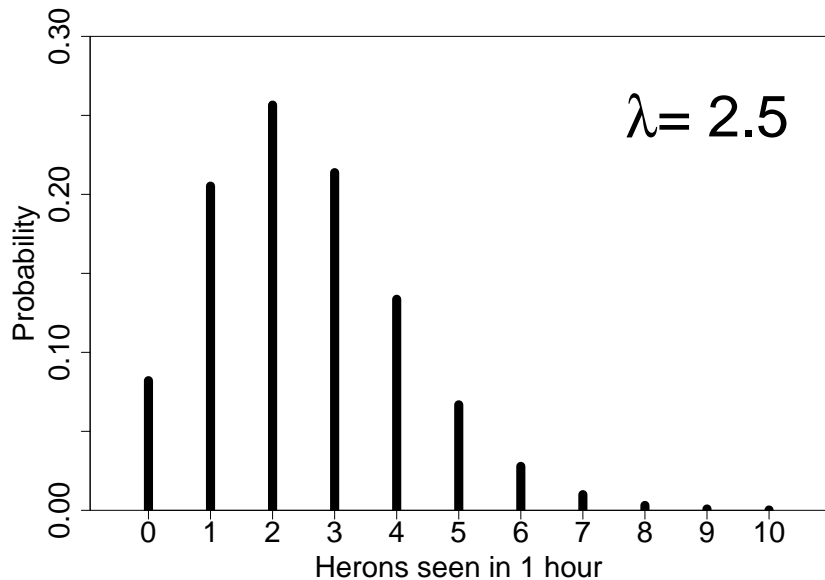
## Probability distributions: Poisson distribution

How many observations over a period of time?

- ▶ Suppose 2.5 herons pass per hour on average
- ▶ Distribution of herons seen in one hour?
- ▶ Sometimes will be more or fewer
- ▶ Distribution has one parameter (rate =  $\lambda$ )
- ▶ If expected to see 2.5 herons,  $\lambda = 2.5$

**Poisson distribution applies to counts,  
so observations natural numbers**

## Probability distributions: Poisson distribution



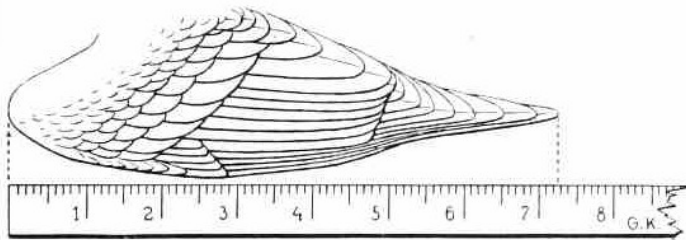
## Count successes or observations

- ▶ Binomial: Yes or No
- ▶ Poisson: Natural number

Describe distribution with a **probability mass function**

## Probability for continuous values

**What if an observation can take any real number?**



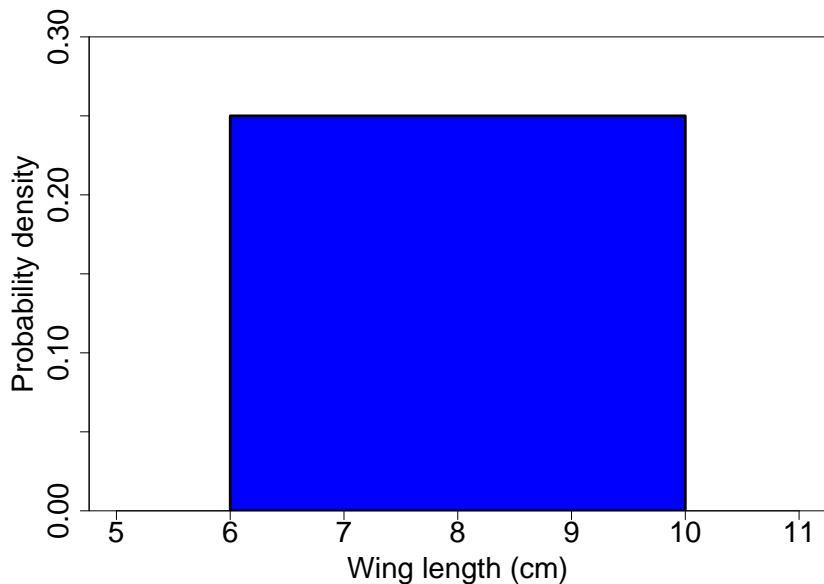
What is the probability of measuring a wing length of 7.28 cm?

<sup>1</sup>Image: Reichenow, A. 1913. (Public domain).

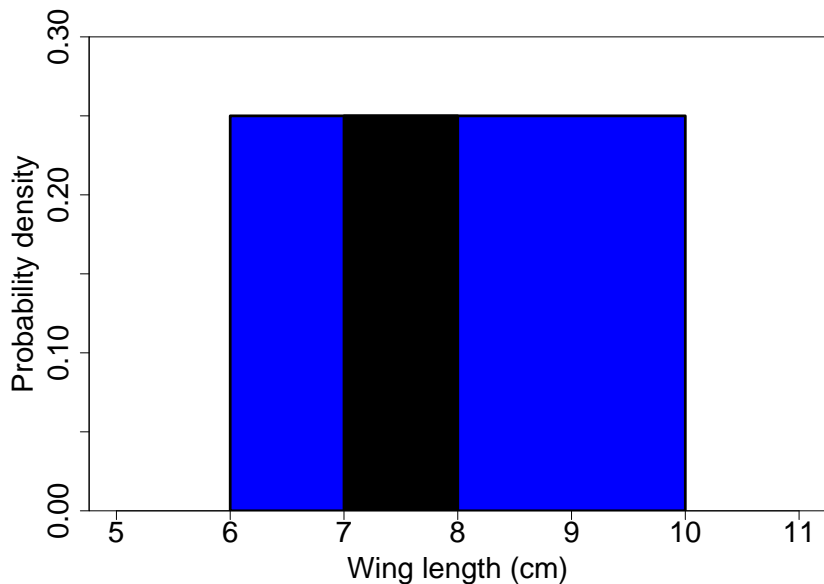
## Probability for continuous values

- ▶ If values are continuous, need to consider a *range* of values
- ▶ Probability wing length between 7 and 8 cm?
- ▶ Describe with a **probability density function**

Uniform distribution (note area sums to 1)



Uniform distribution (note area sums to 1)



Normal distribution (area also sums to 1)

