SCIU4T4: Probability and distributions

Probability: How likely is something to happen?

[1] **Image:** Smith, C. 2013. (Public domain).

Frequentist interpretion of probability

How frequently was the bus late after 120 morning rides?

- Late: 24 times
- Not late: 96 times

Given these observation frequencies, predict the probability that it will be late on any given morning

Frequentist interpretion of probability

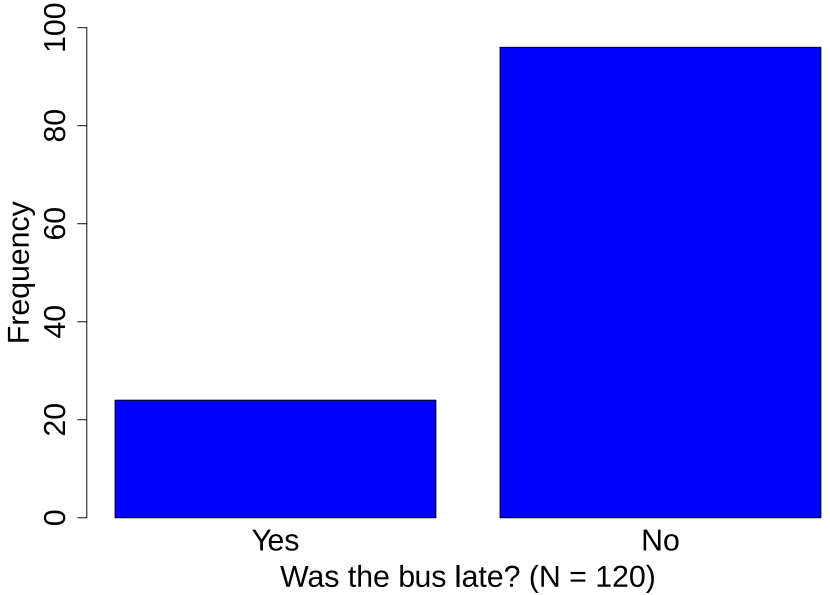
How frequently was the bus late after 120 morning rides?

- Late: 24 times
- Not late: 96 times

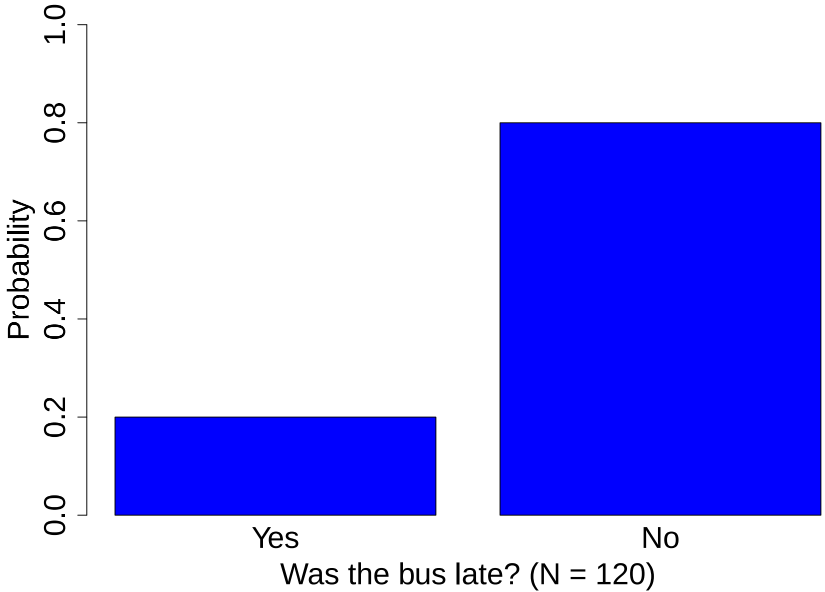
Probability is the long-term *relative* frequency

$$Pr(Late) = \frac{24}{24+96} = \frac{24}{120} = 0.2$$

Frequentist interpretion of probability



Frequentist interpretion of probability



Frequentist interpretion of probability

How frequently was the bus late after 120 morning rides?

- Late: 24 times $\rightarrow$ $Pr(Late) = 0.2$
- Not late: 96 times $\rightarrow$ $Pr(Not\ late) = 0.8$

Values **must** sum to 1:

- Pr(Event) $= 0$: An event *never* occurs
- Pr(Event) $= 1$: An event *always* occurs

Frequentist interpretation of probability

How frequently was the bus late after 120 morning rides?

- Late: 24 times $\rightarrow$ $Pr(Late) = 0.2$
- Not late: 96 times $\rightarrow$ $Pr(Not\ late) = 0.8$

$$\Pr(outcome) = \frac{Times\ outcome\ occurs}{Number\ of\ trials}$$

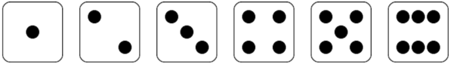
What can we predict?

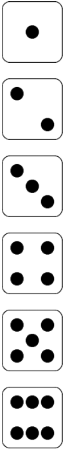
- Extreme climate events
- Genotypes of offspring
- Risk of disease spread
- Population extinction
- Outcomes in games

Games of chance: Probability is a key component

[1]**Image:** Korsalka, Z. 2021. (Public domain).

Rolling two dice; what can happen?

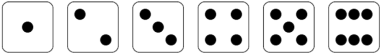


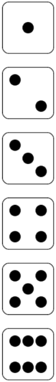


1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

[1] **Image:** All dice images are public domain.

Probability of rolling two 1s



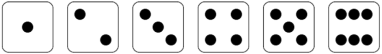


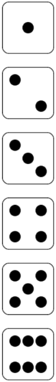
1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

$$\Pr\left(Dice_1 = 1 \; \& \; Dice_2 = 1\right) = \frac{1}{36} = 0.0278$$

[1] **Image:** All dice images are public domain.

Probability of rolling at least one 4



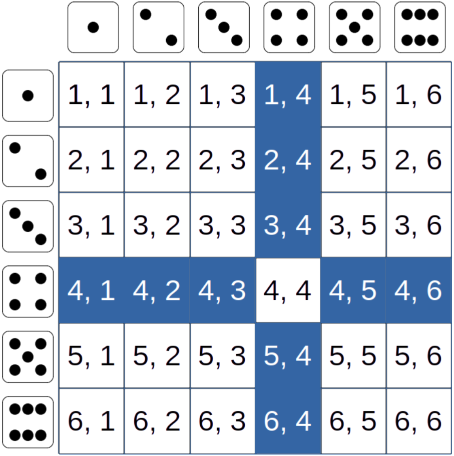


1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

$$\Pr\left(a\ four\right) = \frac{11}{36} = 0.3056$$

[1] **Image:** All dice images are public domain.

Probability of rolling **exactly** one 4



1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

$$\Pr\left(\textit{exactly one four}\right) = \frac{10}{36} = 0.2778$$

[1] **Image:** All dice images are public domain.

Adding or multiplying probabilities

Mutually exclusive: Events cannot happen simultaneously

- Cannot roll 2 values with one dice
- You can roll a 1 **or** a 2, not both

Not mutually exclusive: Events can happen simultaneously

- Can roll 2 values with two dice
- You can roll a 1 **and** another 1

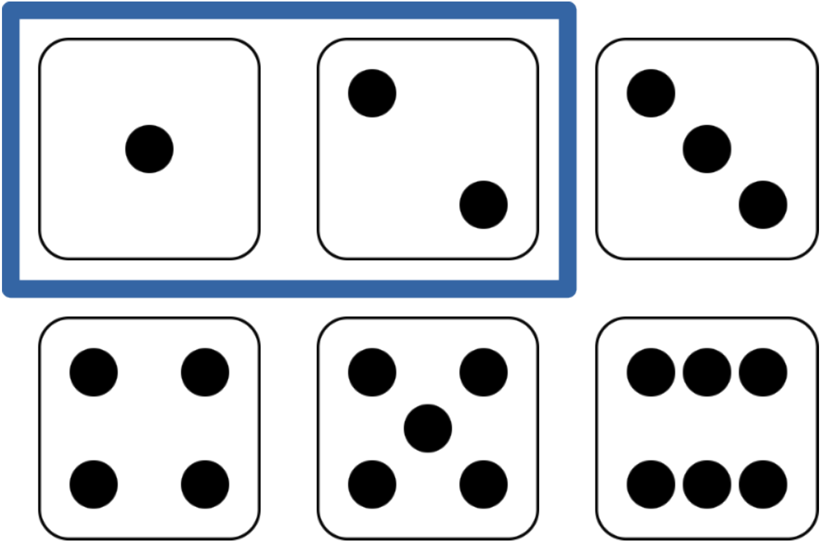
Adding or multiplying probabilities

Mutually exclusive: Events cannot happen simultaneously

- Roll a 1 **or** a 2 with one dice
- $\Pr(\mathit{Roll} = 1 \textit{ or } \mathit{Roll} = 2)$
- $\Pr(\mathit{Roll} = 1) + \Pr(\mathit{Roll} = 2)$

$$\Pr(\textit{one or two}) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = 0.3333$$

Adding or multiplying probabilities



Adding or multiplying probabilities

- Add probabilities when events are mutually exclusive
- Multiply probabilities when events are **not** mutually exclusive

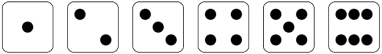
Adding or multiplying probabilities

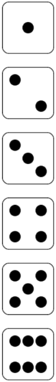
Not mutually exclusive: Events can happen simultaneously

- Can roll 2 values with two dice
- You can roll a 1 **and** another 1
- $\Pr(Roll_1 = 1 \text{ } and \text{ } Roll_2 = 1)$
- $\Pr(Roll_1 = 1) \times \Pr(Roll_2 = 1)$

$$\Pr\left(one \text{ } and \text{ } one\right) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} = 0.0278$$

Probability of rolling a 1 and another 1





1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
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3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

$$\Pr\left(\textit{one and one}\right) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} = 0.0278$$

[1] **Image:** All dice images are public domain.

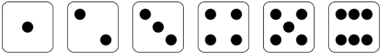
Probability of rolling two 1s **or** two 6s

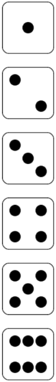
	⚀	⚁	⚂	⚃	⚄	⚅
⚀	**1, 1**	1, 2	1, 3	1, 4	1, 5	1, 6
⚁	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
⚂	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
⚃	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
⚄	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
⚅	6, 1	6, 2	6, 3	6, 4	6, 5	**6, 6**

$$\Pr\left(\textit{two 1s} \mid \textit{two 6s}\right) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36} = 0.0556$$

[1] **Image:** All dice images are public domain.

Probability of rolling a 1 and a 2



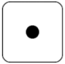


1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

$$\Pr\left(one\ \&\ two\right) = \left(\frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36} = 0.0556$$

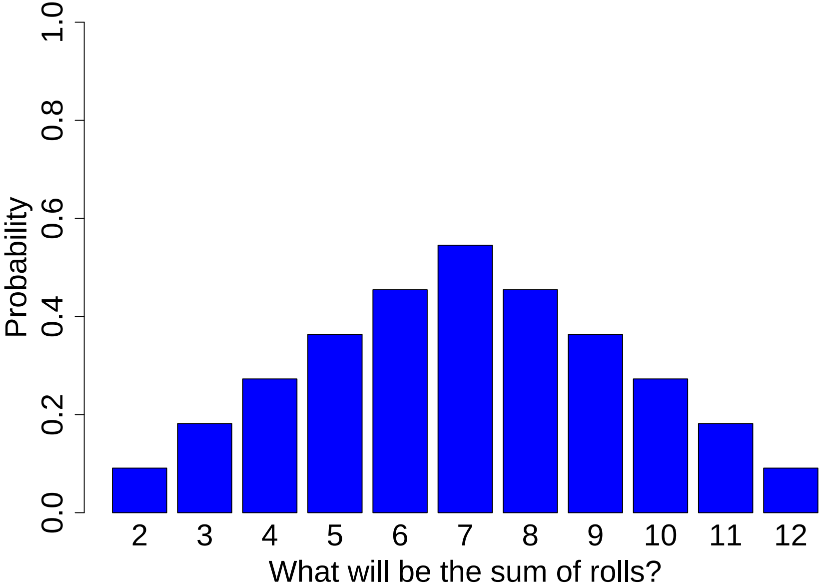
[1] **Image:** All dice images are public domain.

Sum of rolls: what's the distribution?

						
	2	3	4	5	6	7
	3	4	5	6	7	8
	4	5	6	7	8	9
	5	6	7	8	9	10
	6	7	8	9	10	11
	7	8	9	10	11	12

[1]**Image:** All dice images are public domain.

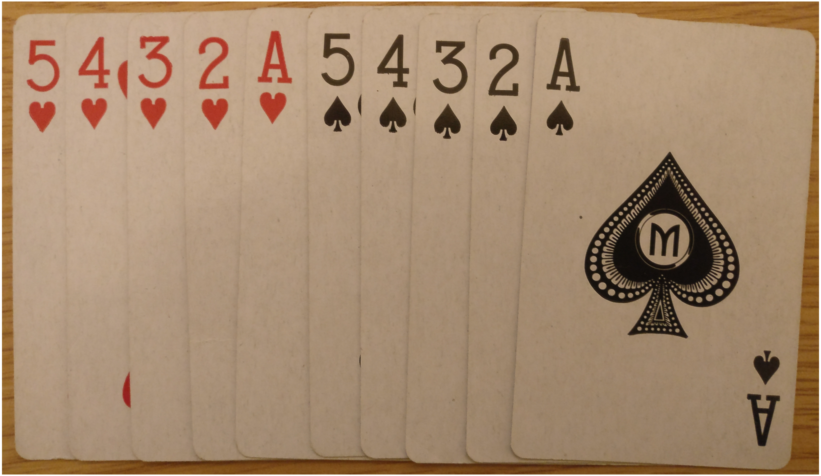
Sum of rolls: what's the distribution?



Sampling with and without replacement

- **With replacement**: Whatever is sampled gets put back before sampling again
- **Without replacement**: Whatever is sampled doesn't get put back before sampling again

Sampling with and without replacement



Pick two cards: probability you sample 2 aces with versus without replacement?

With replacement:

$$\frac{2}{10} \times \frac{2}{10} = 0.04$$

Without replacement

$$\frac{2}{10} \times \frac{1}{9} = 0.0222$$

Event is an individual **trial**

- Successful: $\Pr(X)$
- Unsuccessful: $1 - \Pr(X)$

Successes over N trials

- Binomial distribution
- Coin flips heads

Probability distributions: Binomial distribution

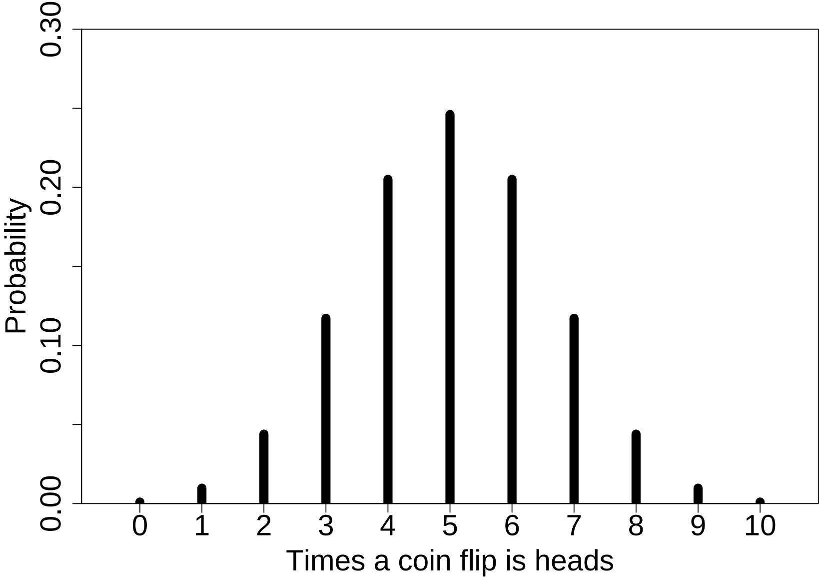


Figure 1: Probability of times a coin flips heads in 10 trials.

Probability distributions: Poisson distribution

How many observations over a period of time?

Also applies to a period of space

[1] **Image:** Public domain.

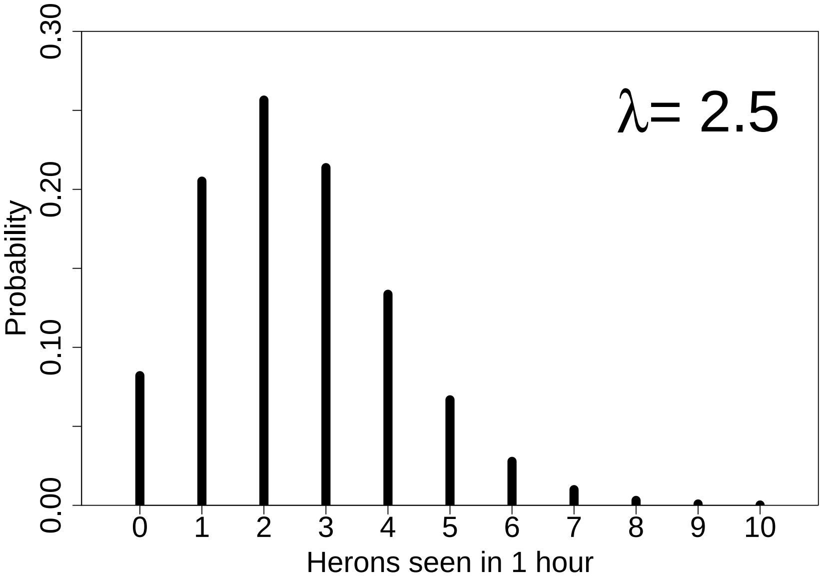
Probability distributions: Poisson distribution

How many observations over a period of time?

- Suppose 2.5 herons pass per hour on average
- Distribution of herons seen in one hour?
- Sometimes will be more or fewer
- Distribution has one parameter (rate $= \lambda$)
- If expected to see 2.5 herons, $\lambda = 2.5$

Poisson distribution applies to counts, so observations natural numbers

Probability distributions: Poisson distribution

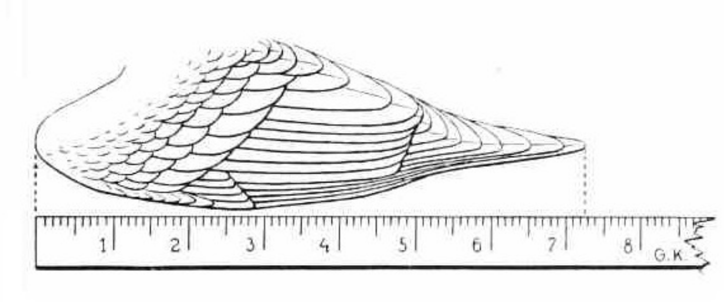


Count successes or observations

- Binomial: Yes or No
- Poisson: Natural number

Describe distribution with a **probability mass function**

What if an observation can take any real number?



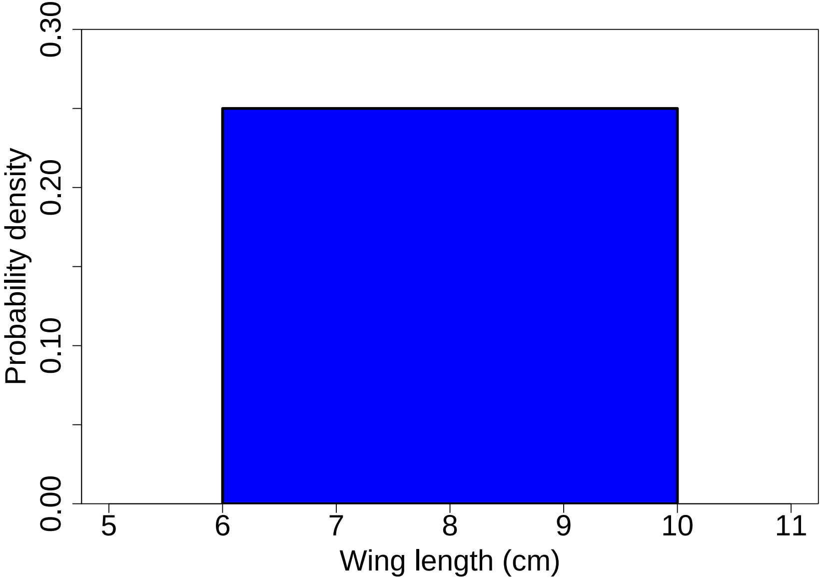
What is the probability of measuring a wing length of 7.28 cm?

[1] **Image:** Reichenow, A. 1913. (Public domain).

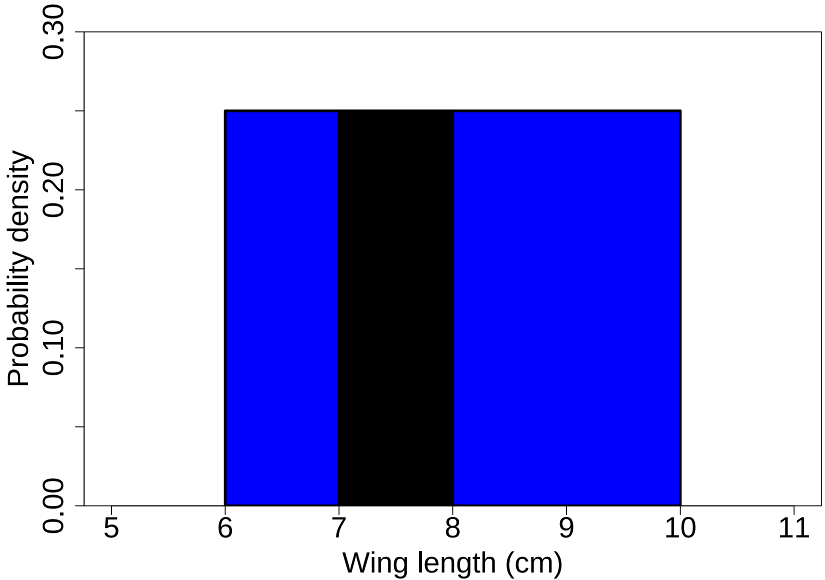
Probability for continuous values

- If values are continuous, need to consider a *range* of values
- Probability wing length between 7 and 8 cm?
- Describe with a **probability density function**

Uniform distribution (note area sums to 1)



Uniform distribution (note area sums to 1)



Normal distribution (area also sums to 1)